p-TEST

I will first state the results. I do not recommend direct memorisation because the results are really easy to obtain. But if you are really out of time, do what you need to.

Theorem. *(p-test) For $a > 0$, we have*

$$\int_a^\infty \frac{1}{x^p} dx \quad \begin{cases} < +\infty, \left(\textit{value is } \frac{a^{1-p}}{p-1}\right), & p > 1 \\ = +\infty, & p \leq 1 \end{cases}$$

$$\int_0^a \frac{1}{x^p} \quad \begin{cases} < +\infty, \left(\textit{value is } \frac{a^{1-p}}{1-p}\right), & p < 1 \\ = +\infty, & p \geq 1 \end{cases}$$

Proof. For the **first** integral,

$$\int_a^\infty \frac{1}{x^p} dx = \lim_{b\to\infty} \int_a^b \frac{1}{x^p} dx = \lim_{b\to\infty} \frac{x^{1-p}}{1-p} \Big|_a^b = \left(\lim_{b\to\infty} \frac{b^{1-p}}{1-p}\right) - \frac{a^{1-p}}{1-p}$$

Clearly, if $1-p > 0$ or equivalently $p < 1$, then $\lim_{b\to\infty} b^{1-p} = +\infty$ and thus $\int_a^\infty \frac{1}{x^p} dx = +\infty$ for $p < 1$.

If $1-p < 0$, or equivalently $p > 1$, $\lim_{b\to\infty} b^{1-p} = \lim_{b\to\infty} \frac{1}{b^{p-1}} = \lim_{b\to\infty} \left(\frac{1}{b}\right)^{p-1} = 0$.

Lastly, for $p = 1$, we have

$$\int_a^\infty \frac{1}{x} dx = \lim_{b\to\infty} \int_a^b \frac{1}{x} dx = \lim_{b\to\infty} \ln|x| \Big|_a^b = \lim_{b\to\infty} \ln b - \ln a = +\infty$$

Therefore, the conclusion for the first integral follows.

For the **second** integral

$$\int_0^a \frac{1}{x^p} dx = \lim_{c\to 0^+} \int_c^a \frac{1}{x^p} dx = \lim_{c\to 0^+} \frac{x^{1-p}}{1-p} \Big|_c^a = \frac{a^{1-p}}{1-p} - \left(\lim_{c\to 0^+} \frac{c^{1-p}}{1-p}\right)$$

Similarly, if $1-p < 0$ or equivalently $p < 1$, then $\lim_{c\to 0^+} c^{1-p} = \lim_{c\to 0^+} \frac{1}{c^{p-1}} = -\infty$ (**why? If you can't figure this out, you should ask me**). Then $\int_0^a \frac{1}{x^p} dx = +\infty$ for $p < 1$.

If $1-p > 0$ or equivalent $p < 1$, then $\lim_{c\to 0^+} c^{1-p} = 0$ obviously. Then $\int_0^a \frac{1}{x^p} dx = \frac{a^{1-p}}{1-p}$ for $p < 1$.

Lastly, for $p = 1$, we have

$$\int_0^a \frac{1}{x} dx = \lim_{c\to 0^+} \int_c^a \frac{1}{x} dx = \lim_{c\to 0^+} \ln|x| \Big|_c^a = \lim_{c\to 0^+} \ln a - \ln c = \ln a - \lim_{c\to 0^+} \ln c$$

One should know that $\lim_{c\to 0^+} \ln c = -\infty$ (**again, why? Ask me if you don't know why.**) Thus,

$$\int_0^a \frac{1}{x} dx = \ln a - \lim_{c\to 0^+} \ln c = +\infty$$

Altogether, the conclusion for the second integral follows. □

Why a? Don't we see $\int_1^\infty \frac{1}{x^p} dx$ in class? Well, the in-class example is a special case. We simply take $a = 1$ here and everything still checks out. Therefore, if you see a problem like

$$\int_{0.5}^\infty \frac{\cos^2(x)}{x^{\frac{3}{2}}} dx$$

Do not panic. Nothing is wrong. As long as that lower bound is NOT 0, p-test conclusion (first one) still holds firmly.

I also suggest that you go through the argument using the above example. Ask me if you need help.